Robust Reconstruction of Complex Networks from Sparse Data

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Reconstructing complex networks from measurable data is a fundamental problem for understanding and controlling collective dynamics of complex networked systems. However, a significant challenge arises when we attempt to decode structural information hidden in limited amounts of data accompanied by noise and in the presence of inaccessible nodes. Here, we develop a general framework for robust reconstruction of complex networks from sparse and noisy data. Specifically, we decompose the task of reconstructing the whole network into recovering local structures centered at each node. Thus, the natural sparsity of complex networks ensures a conversion from the local structure reconstruction into a sparse signal reconstruction problem that can be addressed by using the lasso, a convex optimization method. We apply our method to evolutionary games, transportation, and communication processes taking place in a variety of model and real complex networks, finding that universal high reconstruction accuracy can be achieved from sparse data in spite of noise in time series and missing data of partial nodes. Our approach opens new routes to the network reconstruction problem and has potential applications in a wide range of fields.

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Complex networked systems are common in many fields [1–3]. The need to ascertain collective dynamics of such systems to control them is shared among different scientific communities [4–6]. Much evidence has demonstrated that interaction patterns among dynamical elements captured by a complex network play deterministic roles in collective dynamics [7]. It is thus imperative to study a complex networked system as a whole rather than study each component separately to offer a comprehensive understanding of the whole system [8]. However, we are often incapable of directly accessing network structures; instead, only limited observable data are available [9], raising the need for network reconstruction approaches to uncovering network structures from data. Network reconstruction, the inverse problem, is challenging because structural information is hidden in measurable data in an unknown manner and the solution space of all possible structural configurations is of extremely high dimension. So far a number of approaches have been proposed to address the inverse problem [4,5,9–17]. However, accurate and robust reconstruction of large complex networks is still a challenging problem, especially given limited measurements disturbed by noise and unexpected factors.

In this Letter, we develop a general framework to reconcile the contradiction between the robustness of reconstructing complex networks and limits on our ability to access sufficient amounts of data required by conventional approaches. The key lies in converting the network reconstruction problem into a sparse signal reconstruction problem that can be addressed by exploiting the lasso, a convex optimization algorithm [18,19]. In particular, reconstructing the whole network structure can be achieved by inferring local connections of each node individually via our framework. The natural sparsity of complex networks suggests that on average the number of real connections of a node is much less than the number of all possible connections, i.e., the size of a network. Thus, to identify direct neighbors of a node from the pool of all nodes in a network is analogous to the problem of sparse signal reconstruction. By using the lasso that incorporates both an error control term and an L_1 norm, the neighbors of each node can be reliably identified from a small amount of data that can be much less than the size of a network. The L_1 norm, according to the compressed sensing theory [20], ensures the sparse data requirement while, simultaneously, the error control term ensures the robustness of reconstruction against noise and missing nodes. The whole network can then be assembled by simply matching neighboring sets of all nodes. We will validate our reconstruction framework by considering three representative dynamics, including the ultimatum game [21], transportation [22], and communications [23], taking place in both homogeneous and heterogeneous networks. Our approach opens new routes towards understanding and controlling complex networked systems and has implications for many social, technical, and biological networks.

We articulate our reconstruction framework by taking the ultimatum game (UG) as a representative example. We then apply the framework to the transportation of electrical current and communications via sending data packets.

In the evolutionary UG on networks, each node is occupied by a player. In each round, player i plays the UG twice with each of his or her neighbors, both as a proposer and a responder with strategy (p_i, q_i), where p_i denotes the amount offered to the other player if i proposes and q_i denotes the minimum acceptance level if i

responds [24,25]. The profit of player i obtained in the game with player j is calculated as follows:

$$U_{ij} = \begin{cases} p_j + 1 - p_i & p_i \geq q_j \text{ and } p_j \geq q_i \\ 1 - p_i & p_i \geq q_j \text{ and } p_j < q_i \\ p_j & p_i < q_j \text{ and } p_j \geq q_i \\ 0 & p_i < q_j \text{ and } p_j < q_i \end{cases}, \quad (1)$$

where $p_i, p_j \in [0, 1]$. The payoff g_i of i at a round is the sum of all profits from playing the UG with i's neighbors, i.e., $g_i = \sum_{j\in\Gamma_i} U_{ij}$, where Γ_i denotes the set of i's neighbors. In each round, all participants play the UG with their direct neighbors simultaneously and gain payoffs. Players update their strategies (p, q) in each round by learning from one of their neighbors with the highest payoffs. To be concrete, player i selects the neighbor with the maximum payoff $g_{\max}(t)$ and takes over the neighbor's strategy with probability $W(i \leftarrow \max) = g_{\max}(t)/[g_i(t) + \sum_{j\in\Gamma_i} g_j(t)]$ [26]. To better mimic real situations, random mutation rates are included in each round: all players adjust their (p, q) according to $(p_i(t+1), q_i(t+1)) = (p_i(t) + \delta, q_i(t) + \delta)$, where $\delta \in [-\varepsilon, \varepsilon]$ is a small random number [27]. Without loss of generality, we set $\varepsilon = 0.05$ and $p, q \in [0, 1]$. During the evolution of the UG, we assume that only the time series of $(p_i(t), q_i(t))$ and $g_i(t)$ $(i = 1, \ldots, N)$ are measurable.

The network reconstruction can be initiated from the relationship between strategies $(p_i(t), q_i(t))$ and payoffs $g_i(t)$. Note that $g_i(t) = \sum_{j=1, j\neq i}^{N} a_{ij} U_{ij}$, where $a_{ij} = 1$ if player i and j are connected and $a_{ij} = 0$ otherwise. Moreover, U_{ij} is exclusively determined by the strategies of i and j. These imply that hidden interactions between i and its neighbors can be extracted from the relationship between strategies and payoffs, enabling the inference of i's links based solely on the strategies and payoffs. Necessary information for recovering i's links can be acquired with respect to different time t. Specifically, for M accessible time instances $t_1, \ldots, t_M$, we convert the reconstruction problem into the matrix form $\mathbf{Y}_i = \Phi_i \times \mathbf{X}_i$:

$$\begin{bmatrix} y_i(t_1) \\ y_i(t_2) \\ \vdots \\ y_i(t_M) \end{bmatrix} = \begin{bmatrix} \phi_{i1}(t_1) & \phi_{i2}(t_1) & \ldots & \phi_{iN}(t_1) \\ \phi_{i1}(t_2) & \phi_{i2}(t_2) & \ldots & \phi_{iN}(t_2) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{i1}(t_M) & \phi_{i2}(t_M) & \ldots & \phi_{iN}(t_M) \end{bmatrix} \begin{bmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{iN} \end{bmatrix}, \quad (2)$$

where $\mathbf{Y}_i \in \mathbb{R}^{M\times 1}$ is the payoff vector of i with $y_i(t_\mu) = g_i(t_\mu)$ $(\mu = 1, \ldots, M)$, $\mathbf{X}_i \in \mathbb{R}^{N\times 1}$ is the neighboring vector of i with $x_{ij} = a_{ij}$ $(j = 1, \ldots, N)$, and $\Phi_i \in \mathbb{R}^{M\times N}$ is the virtual-payoff matrix of i with $\phi_{ij}(t_\mu) = U_{ij}(t_\mu)$.

Because $U_{ij}(t)$ is determined by $(p_i(t), q_i(t))$ and $(p_j(t), q_j(t))$ according to Eq. (1), $\mathbf{Y}_i$ and Φ_i can be collected or calculated directly from the time series of strategies and payoffs. Our goal is to reconstruct $\mathbf{X}_i$ from $\mathbf{Y}_i$ and Φ_i. Note that the number of nonzero elements in $\mathbf{X}_i$, i.e., the number of the neighbors of i, is usually much less than the length N of $\mathbf{X}_i$. This indicates that $\mathbf{X}_i$ is sparse, which is ensured by the natural sparsity of complex networks. An intuitive illustration of the reconstruction method is shown in Fig. 1. Thus, the problem of identifying the neighborhood of i is transformed into that of sparse signal reconstruction, which can be addressed by using the lasso.

The lasso is a convex optimization method for solving

$$\min_{\mathbf{X}_i} \left\{ \frac{1}{2M} \|\mathbf{Y}_i - \Phi_i \mathbf{X}_i\|_2^2 + \lambda \|\mathbf{X}_i\|_1 \right\}, \quad (3)$$

where λ is a non-negative regularization parameter [18,19]. The sparsity of the solution is ensured by $\|\mathbf{X}_i\|_1$ in the lasso according to the compressed sensing theory [20]. Meanwhile, the least square term $\|\mathbf{Y}_i - \Phi_i \mathbf{X}_i\|_2^2$ makes the solution more robust against noise in time series and missing data of partial nodes than would the L_1-norm-based optimization method.

The neighborhood of i is given by the reconstructed vector $\mathbf{X}_i$, in which all nonzero elements correspond to direct neighbors of i. In a similar fashion, we construct the reconstruction equations of all nodes, yielding the neighboring sets of all nodes. The whole network can then be assembled by simply matching the neighborhoods of nodes. Because of the sparsity of $\mathbf{X}_i$, it can be reconstructed by using the lasso from a small amount of data that are much less than the length of $\mathbf{X}_i$, i.e., network size N. Although we infer the local structure of each node separately by constructing its own reconstruction equation,

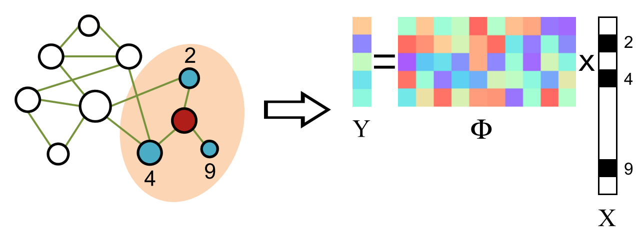

FIG. 1 (color online). Illustration of reconstructing the local structure of a node. For the red node with three neighbors, No. 2, No. 4, and No. 9 in blue, we can establish vector $\mathbf{Y}$ and matrix Φ in the reconstruction form $\mathbf{Y} = \Phi\mathbf{X}$ from data, where vector $\mathbf{X}$ captures the neighbors of the red node. If the reconstruction is accurate, elements in the second, fourth, and ninth rows of $\mathbf{X}$ corresponding to nodes No. 2, No. 4, and No. 9 will be nonzero values (black color), while the other elements are zero (white color). The length of $\mathbf{X}$ is N, which is in general much larger than the average degree of a node, say, three neighbors, ensuring the sparsity of $\mathbf{X}$. In a similar fashion, the local structure of each node can be recovered from relatively small amounts of data compared to the network size by using the lasso. Note that only one set of data is used to reconstruct local structures of different nodes, which ensures the sparse data requirement.

we only use one set of data sampling in time series. This enables a sparse data requirement for recovering the whole network.

We consider current transportation in a network consisting of resistors [22]. The resistance of a resistor between node i and j is denoted by r_{ij}. If i and j are not directly connected by a resistor, $r_{ij} = \infty$. For arbitrary node i, according to Kirchhoff's law, we have

$$\sum_{j=1}^{N} \frac{a_{ij}}{r_{ij}} (V_i - V_j) = I_i, \tag{4}$$

where V_i and V_j are the voltage at i and j and I_i is the total electrical current at i. To better mimic real power networks, alternating current is considered. Specifically, at node i, $V_i = \bar{V} \sin[(\omega + \Delta\omega_i)t]$, where the constant $\bar{V}$ is the voltage peak, ω is the frequency, and $\Delta\omega_i$ is the perturbation. Without loss of generality, we set $\bar{V} = 1$, $\omega = 10^3$, and the random number $\Delta\omega_i \in [0, 20]$. Given voltages at nodes and resistances of links, currents at nodes can be calculated according to Kirchhoff's laws at different time constants. We assume that only voltages and electrical currents at nodes are measurable and our purpose is to reconstruct the resistor network. In an analogy with the UG on networks, based on Eq. (4), we can establish the reconstruction equation $\mathbf{Y}_i = \Phi_i \times \mathbf{X}_i$ with respect to time constants $t_1, \ldots, t_M$, where $y_i(t_\mu) = I_i(t_\mu)$, $x_{ij} = 1/r_{ij}$, and $\phi_{ij}(t_\mu) = V_i(t_\mu) - V_j(t_\mu)$ with $\mu = 1, \ldots, M$ and $j = 1, \ldots, N$. Here, if i and j are connected by a resistor, $x_{ij} = 1/r_{ij}$ is nonzero; otherwise, $x_{ij} = 0$. Thus, the neighboring vector $\mathbf{X}_i$ is sparse and can be reconstructed by using the lasso from a small amount of data. Analogously, the whole network can be recovered by separately reconstructing the neighboring vectors of all nodes.

We propose a simple network model to capture communications in populations via phones, Emails, etc. At each time, individual i may contact one of his or her neighbors j according to probability w_{ij} by sending data packets. If i and j are not connected, $w_{ij} = 0$. In a period, the total incoming flux f_i of i can be described as

$$f_i = \sum_{j=1}^{N} w_{ji} \tilde{f}_j, \tag{5}$$

where $\tilde{f}_j$ is the total outgoing flux from j to its neighbors in the period and $\sum_{i=1}^{N} w_{ji} = 1$. Equation (5) is valid because of the flux conservation in the network. In the real situation, $\tilde{f}_j$ usually fluctuates with time, providing an independent relationship between incoming and outgoing fluxes for constructing the reconstruction equation $\mathbf{Y}_i = \Phi_i \times \mathbf{X}_i$. Here, $y_i(t_\mu) = f_i(t_\mu)$ is the total incoming flux of i at time period t_μ, $\phi_{ij}(t_\mu) = \tilde{f}_j(t_\mu)$ is the total outgoing flux of j at time period t_μ, and $x_{ij} = w_{ji}$ captures connections between i and its neighbors. Given the total incoming and outgoing fluxes of nodes that can be measured without the need of any network information and communication content, we can as well use the lasso to reconstruct the neighboring set of node i and those of the other nodes, such that full reconstruction of the whole network is achieved from sparse data.

We simulate the UG, electrical currents, and communications on both homogeneous and heterogeneous networks, including random [28], small-world [29], and scale-free [30] networks. For the three types of dynamical processes, we record strategies and payoffs of players, voltages and currents, and incoming and outgoing fluxes at nodes at different times, to apply our reconstruction method with respect to different amounts of data (Data $\equiv M/N$, where M is the number of accessible time instances in the time series). Figure 2 shows the results of the UG on small-world networks. For very small amounts of data, e.g., Data $= 0.1$, links are difficult to identify because of the mixture of reconstructed elements in $\mathbf{X}$, whereas for Data $= 0.4$, there is a vast and clear gap between actual links and null connections, ensuring perfect reconstruction [Fig. 2(a)]. Even with strong measurement noise, e.g.,

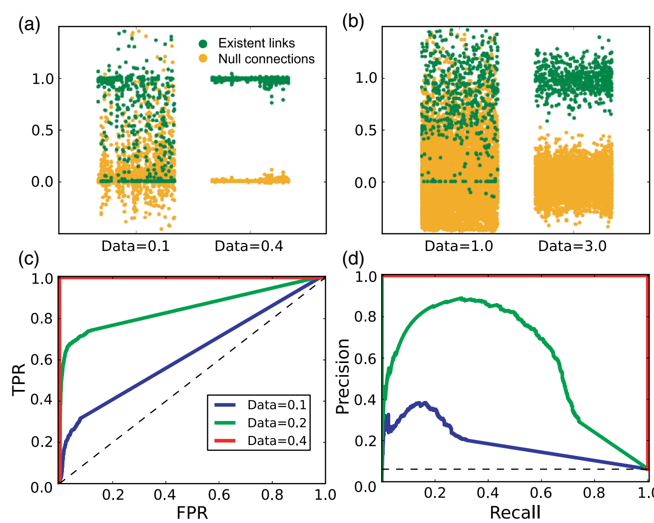

FIG. 2 (color online). Reconstructed values of elements in vector $\mathbf{X}$ for the UG on Watts-Strogatz (WS) small-world networks [29] for different data amounts (a) without measurement noise and (b) with Gaussian noise [$\mathcal{N}(0, 0.3^2)$]. (c) TPR versus FPR and (d) precision versus recall for different data amounts for the UG on WS small-world networks without noise. In (c) and (d), the dashed lines represent the results of completely random guesses. The network size N is 100, and the average degree $\langle k \rangle = 6$. Rewiring probability of small-world networks is 0.3. There are no externally inaccessible nodes. The parameter λ is set to be 10^{-3}. We have tested a wide range of values of λ, finding that optimal reconstruction performance can be achieved in the range $[10^{-4}, 10^{-2}]$ and the reconstruction performance in the range is insensitive to λ. Thus, we set $\lambda = 10^{-3}$ for all reconstructions.

TABLE I. Minimum data for achieving at least 0.95 AUROC and AUPR simultaneously for three types of dynamics, the UG, current transportation, and communications, in combination with three types of networks, random (ER), small world (SW), and scale free (SF). Here, N is the network size, $\langle k \rangle$ is the average degree, σ is the variance of Gaussian noise, and n_m is the proportion of externally inaccessible nodes whose data are missing. Data denote the amount of data divided by network size. The results are obtained by averaging over ten independent realizations. RN denotes resistor network, and CN denotes communication network. More details of the reconstruction performance as a function of data amount for different cases can be found in Ref. [31].

				UG	RN	CN
N	$\langle k \rangle$	σ	n_m		(ER / SW / SF)	
100	6	0	0	0.38 / 0.36 / 0.41	0.28 / 0.25 / 0.32	0.30 / 0.28 / 0.30
	6	0.05	0	0.44 / 0.43 / 0.47	0.29 / 0.26 / 0.37	0.34 / 0.31 / 0.34
	6	0.3	0	1.68 / 1.75 / 1.60	0.32 / 0.29 / 0.38	1.72 / 1.81 / 1.80
	6	0	0.05	0.61 / 0.55 / 0.64	1.61 / 1.65 / 1.60	1.33 / 1.19 / 1.32
	6	0	0.3	2.33 / 2.03 / 2.14	5.74 / 8.51 / 8.50	5.38 / 6.23 / 6.20
	12	0	0	0.46 / 0.47 / 0.52	0.37 / 0 / 35 / 0.42	0.42 / 0.40 / 0.42
	18	0	0	0.53 / 0.53 / 0.58	0.44 / 0.44 / 0.50	0.50 / 0.50 / 0.50
500	6	0	0	0.120 / 0.116 / 0.132	0.094 / 0.080 / 0.120	0.094 / 0.088 / 0.100
1000	6	0	0	0.071 / 0.068 / 0.078	0.058 / 0.049 / 0.079	0.055 / 0.050 / 0.055

$\mathcal{N}(0, 0.3^2)$, by increasing Data, full reconstruction can be still accomplished [Fig. 2(b)]. We use two standard indices, true positive rate (TPR) versus false positive rate (FPR), and precision versus recall to measure quantitatively reconstruction performance [15] (see Ref. [31] for more details). We see that for Data $= 0.4$, both the area under the receiver operating characteristic curve (AUROC) in TPR versus FPR [Fig. 2(c)] and the area under the precision-recall curve (AUPR) in precision versus recall [Fig. 2(d)] equal 1, indicating that links and null connections can be completely distinguished from each other with a certain threshold. Because high reconstruction accuracy can always be achieved, we explore the minimum data for ensuring 0.95 AUROC and AUPR simultaneously for different types of dynamics and networks. As displayed in Table I, with little measurement noise and a small fraction of inaccessible nodes, only a small amount of data is required, especially for large networks, e.g., $N = 1000$. In the presence of strong noise and a large fraction of missing nodes, high accuracy can be still achieved from a relatively larger amount of data. We have also tested our method on several empirical networks (Table II), finding that only sparse data are required for full reconstruction as well. These results demonstrate that our general approach offers robust reconstruction of complex networks from sparse data.

In conclusion, we develop a general framework to reconstruct complex networks with great robustness from sparse data that in general can be much less than network sizes. The key to our method lies in decomposing the task of reconstructing the whole network into inferring local connections of nodes individually. Because of the natural sparsity of complex networks, recovering local structures from time series can be converted into a sparse signal reconstruction problem that can be resolved by using the lasso, in which both the error control term and the L_1 norm jointly enable robust reconstruction from sparse data. Insofar as all local structures are ascertained, the whole network can be assembled by simply matching them. Our method has been validated by the combinations of three representative dynamical processes and a variety of model and real networks with noise and inaccessible nodes. High reconstruction accuracy can be achieved for all cases from relatively small amounts of data.

It is noteworthy that our reconstruction framework is quite flexible and not limited to the networked systems considered here. The crucial issue is to find a certain relationship between local structures and measurable data to construct the reconstruction form $\mathbf{Y} = \Phi\mathbf{X}$. Indeed, there is no general manner to establish the reconstruction form for different networked systems, implying that the application scope of our approach is yet not completely known. Nevertheless, our method could have broad applications in many fields due to its sparse data requirement and its advantages in robustness against noise and missing information. In addition, network reconstruction allows us

TABLE II. Minimum data for achieving at least 0.95 AUROC and AUPR simultaneously for the UG, RN, and CN in combination with several real networks. The variables have the same meanings as in Table I. See Ref. [31] for more details.

	Networks	N	$\langle k \rangle$	Data
UG	Karate	34	4.6	0.69
	Dolphins	62	5.1	0.50
	Netscience	1589	3.5	0.07
RN	IEEE39BUS	39	2.4	0.33
	IEEE118BUS	118	3.0	0.23
	IEEE300BUS	300	2.7	0.10
CN	Football	115	10.7	0.35
	Jazz	198	27.7	0.49
	Email	1133	9.6	0.10

to infer intrinsic nodal dynamics from time series by canceling the influence from neighbors [31], although this is beyond our current scope. Taken together, our approach offers deeper understanding of complex networked systems from observable data and has potential applications in predicting and controlling collective dynamics of complex systems, especially when we encounter explosive growth of data in the information era.

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